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Nowhere-zero 3-flows and Z_3-connectivity of a family of graphs

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ABSTRACT

Tutte conjectured that every 4-edge-connected graph admits a nowhere-zero 3-flow. Jaeger et al. [F. Jaeger, N. Linial, C. Payan, M. Tarsi, Group connectivity of graphs-a nonhomogeneous analogue of Nowhere-zero flow properties, J. Combin. Theory, Ser. B 56 (1992) 165–182] conjectured that every 5-edge-connected graph is Z_3- connected. Let G be a simple connected graph with n vertices. It is proved in this paper that *if* $d(u) + d(v) \geq n$ *for each pair of vertices* u, v *with distance two, then* (1) G *admits a nowhere-zero* 3*-flow if and only if* G *is none of* 7 *excluded graphs;* (2) G *is* Z_3*-connected if and only if* G *is none of* 15 *excluded graphs.* The first theorem strengthens an early result by Fan et al. [G. Fan, C. Zhou. Ore condition and Nowhere-zero 3-flows, SIAM J. Discrete Math., 22 (2008) 288–294] and the second theorem strengthens an early result by Luo, et al. [R. Luo, R. Xu, J.H. Yin, G.X. Yu, Ore-condition and Z_3-connectivity, European J. Combin., 29 (2008) 1587–1595].

1. Introduction

Graphs in this paper are finite and may have multiple edges or loops. Terms and notation not defined here can be found in [1]. Let $G = (V, E)$ be a graph on n vertices. For any $v \in V(G)$, the set of neighbors of v in G is denoted by $N_G(v)$, or simply $N(v)$. Let $N[v] = N(v) \cup \{v\}$ and $d(v) = |N(v)|$. Denote by $\delta(G)$ the minimum degree of G. For two disjoint subgraphs A and B of G, $E(A, B)$ denotes the set of edges with one end-vertex in A and the other end-vertex in B and let $e(A, B) = |E(A, B)|$. For any subset S of $V(G)$, $G - S$ denotes the graph obtained from G by deleting all the vertices of S together with all the edges with at least one end in S. For two distinct vertices u, v of G, the distance of u and v, denoted by $dis_G(u, v)$ is the length of the shortest path between u and v. Thus, if $uv \in E$, then $dis_G(u, v) = 1$ and otherwise $dis_G(u, v) \geq 2$. For any $u \in V(G)$ and for each $2 \leq k < n$, define

$$N_k(u) = \{v | dis_G(v, u) = k, v \in V(G)\}.$$

Given a subset S of $V(G)$, $dis_G(u, S) = 2$ means $dis_G(u, v) \geq 2$ for any $v \in S$ and there exists at least one $v \in S$ such that $dis_G(u, v) = 2$.

The complete graph on n vertices is denoted by K_n, and K_n^- is obtained from K_n by deleting an edge. $K_{3,n-3}^+$ denotes the simple graph obtained from the complete bipartite graph $K_{3,n-3}$ by adding an edge between two vertices of degree $n - 3$. The n-circuit, denoted by C_n, is a circuit on n vertices. The wheel W_k is the graph obtained from a k-circuit by adding a new vertex and joining it to every vertex of the k-circuit. W_k is odd (even) if k is odd (even). For a technical reason, a single edge is regarded as 1-circuit, and thus W_1 is a triangle, called the trivial wheel. Let H be a connected subgraph of G. G/H denotes the graph obtained from G by contracting all the edges of H and deleting all the resulting loops.

For an orientation D of G and a vertex $v \in V(G)$, we use $E^+(v)$ ($E^-(v)$) to denote the set of edges with tails (heads) at v. Let A be an abelian group and $A^* = A - \{0\}$. Define

$$F(D, A) = \{f : E(D) \to A\} \quad \text{and} \quad F^*(D, A) = \{f : E(D) \to A^*\}.$$

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For each $f \in F(D, A)$, the boundary of f is the function $\partial f : V(D) \to A$ defined by

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e)$$

for each vertex $v \in V(D)$. We say that G is A-connected if G has an orientation D such that for every function $b : V(G) \to A$ with $\sum_{v \in V(G)} b(v) = 0$, there exists a function $f \in F^*(D, A)$ with boundary $\partial f = b$. An A-nowhere-zero-flow in G is a function $f \in F^*(D, A)$ such that $\partial f = 0$.

Since the existence of a nowhere-zero mapping with a specified boundary depends only on the underlying undirected graph and not on the orientation of the edges, we only talk about group connectivity of undirected graphs.

The concept of A-connectivity was introduced by Jaeger et al. [7] as a generalization of nowhere-zero flows. The following two conjectures are well known.

Conjecture 1.1 (*Tutte, Unsolved Problem 48 in [1]*). *Every 4-edge connected graph admits a nowhere-zero Z_3-flow.*

Conjecture 1.2 (*Jaeger et al. [7]*). *Every 5-edge connected graph is Z_3-connected.*

A simple graph G of order n satisfies the Ore-condition [10] if $d(x)+d(y) \geq n$ for each pair of non-adjacent vertices x and y. Fan et al. [5] investigated the degree sum for nowhere-zero Z_3-flows. They also studied the Ore-condition for nowhere-zero Z_3-flows [6]. Luo et al. [9] considered the Ore-condition and Z_3-connectivity. The results are listed as follows.

Theorem 1.3 (*Fan et al. [5]*). *Let G be a 2-edge-connected simple graph on n vertices. If $d(x) + d(y) \geq n$ for each $xy \in E$, then G has no nowhere-zero 3-flow if and only if G is $K^+_{3,n-3}$ or one of G_4, G_5, G_7, G_{11} and G_{12} in Fig. 1.*

Theorem 1.4 (*Fan et al. [5]*). *Let G be a simple graph on n vertices. If $d(x)+d(y) \geq n+2$ for each $xy \in E$, then G is Z_3-connected if and only if G is not K_4.*

Theorem 1.5 (*Fan et al. [6]*). *Let G be a simple graph on n vertices, $n \geq 3$. If G satisfies the Ore-condition, then G has no nowhere-zero 3-flow if and only if G is one of $G_4, G_5, G_7, G_9, G_{11}$ and G_{12} in Fig. 1.*

Theorem 1.6 (*Luo et al. [9]*). *A simple graph G satisfying the Ore-condition with at least 3 vertices is not Z_3- connected if and only if G is one of the first 12 graphs in Fig. 1.*

In this paper, we prove the following theorem.

Theorem 1.7. *Let G be a simple connected graph with $n \geq 3$ vertices. If $d(u) + d(v) \geq n$ for each pair of vertices $u, v \in V(G)$ with $dis_G(u, v) = 2$, then G is Z_3-connected if and only if G is none of the 15 graphs illustrated in Fig. 1.*

If a graph G is Z_3-connected, then G admits a nowhere-zero Z_3-flow. The Ore-condition implies the degree condition in Theorem 1.7, by checking the degree of each graph in Fig. 1, we see that Theorems 1.5 and 1.6 are corollaries of Theorem 1.7. Furthermore, it is not difficult to check that each of $G_1, G_2, G_3, G_6, G_8, G_{10}, G_{13}, G_{14}$ in Fig. 1 admits a nowhere-zero 3-flow, and each of the remaining graphs in Fig. 1 has no nowhere-zero 3-flow.

Corollary 1.8. *Suppose G is a simple connected graph of order n. If $d(u) + d(v) \geq n$ for each pair of vertices $u, v \in V(G)$ with $dis_G(u, v) = 2$, then G admits a nowhere-zero 3-flow if and only if G is none of $\{G_4, G_5, G_7, G_9, G_{11}, G_{12}, G_{15}\}$ listed in Fig. 1.*

Finally, the following corollary is considered as a partial result to Conjectures 1.1 and 1.2.

Corollary 1.9. *Suppose G is a simple connected graph of order n with $\delta(G) \geq 4$. If $d(u) + d(v) \geq n$ for each pair of vertices $u, v \in V(G)$ with $dis_G(u, v) = 2$, then G is Z_3-connected, and so G admits a nowhere-zero Z_3-flow.*

Recently, Zhang et al. considered degree sum condition for Z_3-connectivity in graphs. They proved if a 2-edge-connected simple graph G satisfies $d(x) + d(y) \geq n$ for each $xy \in E(G)$, then G is either Z_3-connected or one of some exceptional graphs [11].

The rest of the paper is organized as follows: In Section 2, former related results are presented and in Section 3, some lemmas for proving the theorem are given. In Section 4, the main theorem is proved.

2. Known results

A graph G is triangularly connected if for every pair of edges $e_1, e_2 \in E$, there exists a sequence of circuits $C_1, C_2, \ldots, C_k$ such that $e_1 \in E(C_1)$ and $e_2 \in E(C_k)$, $|E(C_i)| \leq 3$ for $1 \leq i \leq k$, and such that $E(C_j) \cap E(C_{j+1}) \neq \emptyset$ for $1 \leq i \leq k - 1$. Let H_1

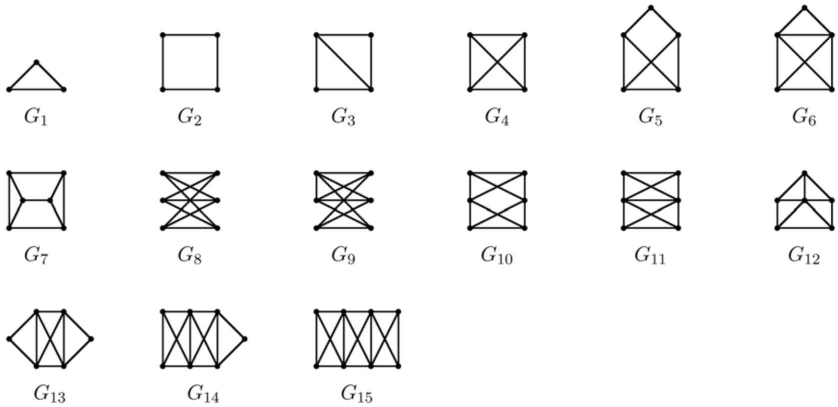

Fig. 1. Non-Z_3-connected graphs.

and H_2 be two subgraphs of a graph G. We say that G is the 2-sum of H_1 and H_2, denoted by $H_1 \oplus H_2$, if $E(H_1) \cup E(H_2) = E(G)$, $|V(H_1) \cap V(H_2)| = 2$ and $|E(H_1) \cap E(H_2)| = 1$.

Lemma 2.1 (Lai [8]). *Let A be an abelian group. If H is a subgraph of G and if both H and G/H are A-connected, then G is A-connected.*

Lemma 2.2 ([2–4,7,8]). *Let A be an abelian group with $|A| \geq 3$. The following results are known:*

(1) *K_n and K_n^- are A-connected if $n \geq 5$.*
(2) *C_n is A-connected if and only if $|A| \geq n+1$.*
(3) *$K_{m,n}$ is A-connected for $m \geq n \geq 4$ and $K_{m,3}$ is not Z_3-connected.*
(4) *Each even wheel is Z_3-connected and each odd wheel is not.*
(5) *If G is not A-connected, then any spanning subgraph of G is not A-connected.*

Lemma 2.3 (Fan et al. [4]). *Let G be a triangularly connected graph. Then G is A-connected for all abelian group A with $|A| \geq 3$ if and only if $G \neq H_1 \oplus H_2 \oplus \cdots \oplus H_k$, where H_i is an odd wheel (including a triangle) for $1 \leq i \leq k$.*

Lemma 2.4 (DeVos et al. [3]). *Let G be a loopless triangularly connected graph with $\delta(G) \geq 4$, then G is Z_3-connected.*

Let $G = (V, E)$ be a graph and let u, v, w be three vertices of G with $uv, uw \in E$. Define $G_{[uv,uw]}$ to be the graph $G \cup \{vw\} \setminus \{uv, uw\}$.

Lemma 2.5 (Chen et al. [2]). *Let A be an abelian group. Let $G = (V, E)$ be a graph and let u, v, w be three vertices of G with degree $d(u) \geq 4$ and $uv, uw \in E$. If $G_{[uv,uw]}$ is A-connected, then so is G.*

3. Lemmas

Let S be a subset of $V(G)$. For simplicity, in the rest of paper, the subgraph induced by S is denoted by $\langle S \rangle$. By Lemma 2.1 and (2) of Lemma 2.2, the following observation holds.

Observation 3.1. *Let H be a subgraph of G. If H is Z_3-connected and for each $v \in V(G) \setminus V(H)$, $e(v, H) \geq 2$, then G is Z_3-connected.*

Lemma 3.2. *No graph in Fig. 1 is Z_3-connected.*

Proof. By Theorem 1.6, each of the first 12 graphs in Fig. 1 is not Z_3-connected. Each of the rest three graphs $G_{13} \sim G_{15}$ can be written as $G_{13} = W_1 \oplus W_3 \oplus W_1$, $G_{14} = W_3 \oplus W_3 \oplus W_1$ and $G_{15} = W_3 \oplus W_3 \oplus W_3$. By Lemma 2.3, the conclusion follows. $\square$

Lemma 3.3 (Luo et al. [9]). *Each graph in Fig. 2 is Z_3-connected.*

Lemma 3.4. *Let G be a simple connected graph on $n \geq 6$ vertices and A be an abelian group with $|A| \geq 3$. If $d(u) + d(v) \geq n$ for each pair of vertices u, v with $dis_G(u, v) = 2$, then G is A-connected or G is one of G_7, G_8 in Fig. 1, or G contains K_4^- as a subgraph.*

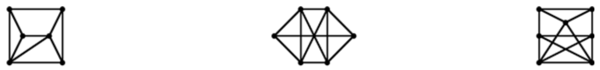

Fig. 2. Z_3-connected graphs.

Proof. We consider the following two cases according to the value of the minimum degree of G.

Case 1. $\delta(G) \le \frac{n-1}{2}$.

Let x be a vertex with the minimum degree and $N(x) = \{x_1, x_2, \ldots, x_s\}$, where $s = \delta(G)$. Then the degree condition of the lemma implies $|N(x)| \ge 2$. Otherwise, $N(x) = \{x_1\}$ and let $y \ne x$ be a neighbor of x_1. Since $dis_G(x, y) = 2$, we have $d(y) = n - 1$. This contradicts $xy \notin E$. If $\langle N(x)\rangle$ contains a path of length two, say $x_1x_2x_3$, then the subgraph induced by $\{x, x_1, x_2, x_3\}$ contains a K_4^-, we are done. So assume that all edges (if there exist) in $\langle N(x)\rangle$ are independent. Let $M = V(G) \setminus (N[x] \cup N_2(x))$.

Claim 1. Either G contains K_4^- as a subgraph or $|N_2(x)| \ge \frac{n}{2} - 1$.

Proof. If there is no edge in $\langle N(x)\rangle$, then any pair of vertices in the subgraph has distance two and so $d(x_1) + d(x_2) \ge n$. This implies $|N_2(x)| \ge \frac{n}{2} - 1$.

Suppose that there exists at least one edge in $\langle N(x)\rangle$, say $x_1x_2 \in E$. If $\delta(G) = 2$, then $d(y) \ge n - 2$ for any $y \in N_2(x)$, which means y is adjacent to each vertex of $V(G) \setminus \{x\}$. Then K_4^- is a subgraph induced by $\{x, x_1, x_2, y\}$. Hence $\delta(G) \ge 3$. Suppose there is another edge in $\langle N(x)\rangle$, say $x_3x_4 \in E$. In this case $d(x_1) + d(x_3) \ge n$ and $d(x_2) + d(x_4) \ge n$, say the former holds. Since x_1x_2 and x_3x_4 are two independent edges of $\langle N(x)\rangle$, we have $|N(x_1) \cap N(x_2)| \ge n - (n - 2) = 2$. Hence K_4^- is a subgraph induced by x, x_1, x_2 and a vertex in $N(x_1) \cap N(x_2) \setminus \{x\}$.

Now we may assume that there is only one edge x_1x_2 in $\langle N(x)\rangle$ and further assume $N(x_1) \cap N(x_2) = \{x\}$. This implies $d(x_1) + d(x_2) \le n$. Since $dis_G(x_i, x_3) = 2$ for each $i = 1, 2$, we have $d(x_1) + d(x_2) + 2d(x_3) \ge 2n$, and so $d(x_3) \ge \frac{n}{2}$. Thus $|N_2(x)| \ge \frac{n}{2} - 1$. □

It follows by Claim 1 that $|N(x) \cup M| \le \frac{n}{2}$. As $d(y) \ge n - \delta(G) \ge \frac{n+1}{2}$ for any $y \in N_2(x)$, there is an edge y_1y_2 in $\langle N_2(x)\rangle$. Therefore, $|N(y_1) \cap N(y_2)| \ge n + 1 - (n - 1) = 2$. Let $w_1, w_2 \in N(y_1) \cap N(y_2)$. Then K_4^- is a subgraph induced by $\{y_1, y_2, w_1, w_2\}$.

Case 2. $\delta(G) \ge \frac{n}{2}$.

In this case, $|E(G)| \ge \frac{n^2}{4}$. By the Turán's Theorem, either G contains a triangle or $G \cong K_{m,m}$, where $m = \frac{n}{2}$. If $G \cong K_{3,3}$, then it is G_8. If $G \cong K_{m,m}$ and $m \ge 4$, then by Lemma 2.2, G is A-connected. Let G contains a triangle $T = v_1v_2v_3v_1$. For any $u \in S = V(G) \setminus V(T)$, if u has two neighbors in T, then G contains a K_4^-. Otherwise, we have

$$\frac{3n}{2} \le d(v_1) + d(v_2) + d(v_3) \le (n - 3) + 6 = n + 3.$$

This implies $n = 6$. We assume $u_1, u_2, u_3 \in S$ and $E(T, S) = \{u_1v_1, u_2v_2, u_3v_3\}$. Since $\delta(G) = 3$, $d_S(u_i) = 2$ for each $i = 1, 2, 3$. It is easy to see that in this case G is G_7. The lemma is proved. □

Lemma 3.5. *Let G be a simple connected graph on $n \ge 6$ vertices with $\delta(G) \ge 3$. Suppose $d(u) + d(v) \ge n$ for each pair of vertices u, v with $dis_G(u, v) = 2$. If there are at least two vertices of degree three in G, then either G is Z_3-connected or G is one of $G_7 \sim G_{15}$ in Fig. 1.*

Proof. Let x and y be two vertices of degree three. If $dis_G(x, y) = 2$, then $n = 6$ and $|N(x) \cap N(y)| \ge 2$. First let $N(y) = N(x)$ and the neighbors of x and y be x_1, x_2, x_3. Let z be the remaining vertex of G. Then G must contain G_8 as a spanning subgraph. If G is not G_8 or G_9, then there are at least two edges in $\langle N(x)\rangle$. it follows that $G \setminus \{x\}$ contains a W_4. By Lemma 2.2 and Observation 3.1, G is Z_3-connected.

So assume that $N(x) \cap N(y) = \{x_1, x_2\}$, $z_1 \in N(x) \setminus N(y)$ and $z_2 \in N(y) \setminus N(x)$. Since $\delta(G) \ge 3$, either

$$z_1z_2 \in E \quad \text{and} \quad e(z_i, \{x_1, x_2\}) \ge 1 \quad \text{for each } i = 1, 2, \tag{1}$$

or

$$z_1z_2 \notin E \quad \text{and} \quad e(z_i, \{x_1, x_2\}) = 2 \quad \text{for each } i = 1, 2. \tag{2}$$

Suppose (1) holds. If $e(x_i, \{z_1, z_2\}) \ge 1$ for each $i = 1, 2$, G must contain a copy of G_7 as its spanning subgraph. If G is not G_7, then $G \supseteq G_1^*$ (in Fig. 2). By Lemma 3.3, G is Z_3-connected. So assume $e(x_1, \{z_1, z_2\}) = 0$. Since $\delta(G) \ge 3$, we have $x_1x_2 \in E$ and $e(x_2, \{z_1, z_2\}) = 2$. So G is G_{12}. Hence (2) holds. Then G is G_{10} if $x_1x_2 \notin E$ and G is G_{11} otherwise.

So $dis_G(x, y) \ne 2$. We consider the following two cases $xy \in E$ or $dis_G(x, y) \ge 3$.

Case 1. $xy \in E$.

Let $N(x) = \{x_1, x_2, y\}$ and $N(y) = \{y_1, y_2, x\}$. For simplicity, write $N_0 = \{x, y\}$. We further let $N_1 = N(x) \cup N(y) \setminus N_0$, $N_2 = N_2(x) \cup N_2(y) \setminus (N_0 \cup N_1)$, and $M = V(G) \setminus \bigcup_{j=0}^{2} N_j$.

Since $dis_G(v, \{x, y\}) = 2$ for any $v \in N_2$, we have $d(v) \geq n - 3$. That is

$$v \text{ is adjacent to each vertex in } V(G) \setminus \{x, y\} \quad \text{for any } v \in N_2. \tag{3}$$

So the vertices in N_2 together with any two vertices $u, u' \in N_1$, induces $G' \cong K_m$ or $G' \cong K_m^-$. Furthermore, we have $e(z, G') \geq |N_2|$ for any vertex $z \in M \cup (N_1 \setminus \{u, u'\})$. If $|N_2| \geq 3$, then $m \geq 5$. By (1) of Lemma 2.2, G' is Z_3-connected. By Observation 3.1, $G'' = \langle V(G') \cup M \cup (N_1 \setminus \{u, u'\}) \rangle$ is Z_3-connected. Since $e(x, G'') = 2$ and $e(y, G'') = 2$, again by Lemma 2.2 and Observation 3.1, $G = \langle V(G'') \cup \{x, y\} \rangle$ is Z_3-connected. So assume $|N_2| \leq 2$.

Case 1.1. $M \neq \emptyset$.

Claim 1. If $M \neq \emptyset$, then $|N_2| \geq 3 + |N_1| - e(u, N_1 - u) - e(u, \{x, y\})$ for any $u \in N_1$.

Proof. By (3), we have $dis_G(u, z) = 2$ for any $u \in N_1$ and any $z \in M$. By the degree condition and $d(u) = e(u, N_2) + e(u, N_1 - u) + e(u, \{x, y\})$, we have

$$n - |N_2| - e(u, N_1 - u) - e(u, \{x, y\}) \leq d(z) \leq |N_2| + |M| - 1.$$

It follows by $n - |M| = 2 + |N_1| + |N_2|$ that $|N_2| \geq 3 + |N_1| - e(u, N_1 - u) - e(u, \{x, y\})$. □

Since $e(u, N_1 - u) + e(u, \{x, y\}) \leq |N_1| + 1$ for any $u \in N_1$, Claim 1 implies $|N_2| \geq 2$. Recall that $|N_2| \leq 2$, we have $|N_2| = 2$. Suppose $|N_1| \geq 3$. Then there exists $u \in N_1$ such that $e(u, \{x, y\}) = 1$. By Claim 1, we have $|N_2| \geq 3$, a contradiction. So $|N_1| = 2$. Without loss of generality, assume $N(x) = N(y) = \{x_1, x_2\}$. Since $|N_2| = 2$, by Claim 1, we have $x_1x_2 \in E$. It follows that $d(x_1) = 5$. By (3), $dis_G(z, N_1) = 2$ for each $z \in M$. So $d(z) \geq n - 5$. This implies that $\langle M \rangle$ contains a complete subgraph. If $|M| = 1$, then G is G_{14}; If $|M| = 2$, then G is G_{15}; If $|M| \geq 3$, then $G' = \langle N_2 \cup M \rangle$ is K_m with $m \geq 5$. By Lemma 2.2 and Observation 3.1, $G'' = \langle V(G') \cup N_1 \rangle$ is Z_3-connected, so is G.

Case 1.2. $M = \emptyset$.

In this case, $V(G) = N_2 \cup N_1 \cup \{x, y\}$. Since $|N_2| \leq 2$ and $|N_1| \leq 4$, we have $6 \leq n \leq 8$. If $|N_1| = 2$, then $|N_2| = 2$. It is easy to see that G is G_{11} or G_{10}. If $|N_1| = 3$, assume that $x_2 = y_2 \in N(x) \cap N(y)$, then $dis_G(x_1, y) = dis_G(y_1, x) = 2$. Since $d(x_1) \geq n - 3$ and $d(y_1) \geq n - 3$, either $x_1y_1 \in E$ or there is a path $x_1x_2y_1$. If N_1 induces a triangle, then W_4 is a subgraph induced by $N_1 \cup \{x, y\}$. It follows by Lemma 2.2 and Observation 3.1 that G is Z_3-connected. If N_1 induces a path $x_1x_2y_1$ and $|N_2| = 2$, then $\langle N_1 \cup N_2 \rangle \cong K_5^-$, and so G is Z_3-connected. If $|N_2| = 1$, then G is G_{12} in Fig. 1. If $x_1y_1 \in E$ and $|N_2| = 2$, then G contains G_3^* in Fig. 2 as a spanning subgraph. By Lemma 3.3, G is Z_3-connected. If $x_1y_1 \in E$ and $|N_2| = 1$, then G contains G_7 in Fig. 1 as a spanning subgraph. If it is not G_7, then G contains G_1^* as a spanning subgraph. So by Lemma 3.3, G is Z_3-connected.

If $|N_1| = 4$, then $d(u) \geq n - 3$ for each $u \in N_1$, since $dis_G(u, x) = 2$ for $u \notin N(x)$, or $dis_G(u, y) = 2$ for $u \notin N(y)$. Therefore, $e(u, N_1) \geq n - 3 - 1 - |N_2| = 2$. This implies that $\langle N_1 \rangle$ contains a spanning circuit. If $N_2 \neq \emptyset$, then $\langle N_1 \cup \{v\} \rangle \supseteq W_4$, where $v \in N_2$. Since $e(u, G') \geq 2$ for any $u \in V(G) \setminus (N_1 \cup \{v\})$, again by Lemma 2.2 and Observation 3.1, G is Z_3-connected. If $N_2 = \emptyset$, then G is one of G_7, G_1^* and G_2^*. By Lemma 3.3, the conclusion follows.

Case 2. $dis_G(x, y) \geq 3$.

Let $N(x) = \{x_1, x_2, x_3\}$, $N(y) = \{y_1, y_2, y_3\}$ and $M = V(G) - (N(x) \cup N_2(x) \cup \{x, y\})$. Since $d(x_i) + d(z) \geq n$ for any $z \in N_3(x)$ with $dis_G(z, x_i) = 2$, and $d(x_i) \leq 1 + d_{N(x)}(x_i) + |N_2(x)|$, we have $d(z) \geq n - 1 - d_{N(x)}(x_i) - |N_2(x)|$. It follows by $d(z) \leq n - 5$ that the following claim holds:

Claim 2. $|N_2(x)| \geq 4 - d_{N(x)}(x_i)$ for each x_i with $dis_G(x_i, N_3(x)) = 2$.

Case 2.1. $N_2(x) \cap N(y) = \emptyset$.

By the degree condition, $d(u) \geq n - 3$ for any $u \in N_2(x)$. That is, u is adjacent to each of vertices in $V(G) \setminus \{x, y\}$. If there exists x_i such that $d_{N(x)}(x_i) = 2$, then let $G' = \langle N(x) \cup N_2(x) \rangle$, and otherwise, let $G' = \langle N_2(x) \cup \{x_1, x_2\} \rangle$. So $G' \cong K_m$ or $G' \cong K_m^-$. By Claim 2, $m \geq 5$. Since for any vertex of $u \in V(G) \setminus (V(G') \cup \{x, y\})$, we have $e(u, G') \geq |N_2| \geq 2$ and $d(x) = d(y) = 3$, by Lemma 2.2 and recursively using Observation 3.1, G is Z_3-connected.

Case 2.2. $|N_2(x) \cap N(y)| \geq 1$.

If $|N_2(x) \setminus N(y)| \geq 2$, then $d(u) \geq n - 3$ for any $u \in N_2(x) \setminus N(y)$, and $e(v, N(x)) \geq 2$ for any $v \in N_2(x) \cap N(y)$. Let $G' = \langle (N_2(x) \setminus N(y)) \cup \{x_1, x_2, v\} \rangle$, where v is any vertex of $N_2(x) \cap N(y)$ with $vx_1, vx_2 \in E$. Therefore, $G' \cong K_m$ or $G' \cong K_m^-$ with $m \geq 5$. By the similar argument as in Case 2.1, G is Z_3-connected. So assume $|N_2(x) \setminus N(y)| \leq 1$.

Claim 3. $dis_G(x_i, y) = 2$, and so $d(x_i) = n - 3$ for each $i = 1, 2, 3$.

Proof. Otherwise, as each vertex in $N_2(x) \cap N(y)$ has at most one nonadjacent vertex in $V(G) \setminus \{x\}$, we may assume $dis_G(x_3, y) \geq 3$. Then $x_3y_i \notin E$ for each vertex $y_i \in N_2(x) \cap N(y)$. If $|N_2(x)| \geq 3$, then $G' = \langle N_2(x) \cup \{x_1, x_2\} \rangle$ is K_m or K_m^- with $m \geq 5$. We may further check that $e(u, G') \geq 2$ for each $u \in V(G) \setminus V(G')$, by the similar argument above, G is Z_3-connected. Therefore, $|N_2(x)| \leq 2$. It follows from Claim 2 that $d_{N(x)}(x_i) = 2$ for each $i = 1, 2, 3$ and $|N_2(x)| = 2$, and so $N(x)$ induces a triangle. It is easy to see that $G' = \langle N(x) \cup N_2(x) \rangle$ contains K_5^- as its spanning subgraph. Since $e(u, G') \geq 2$ for each $u \in V(G) \setminus V(G') \cup \{x, y\}$, and $d(x) = d(y) = 3$, we have that G is Z_3-connected. □

By Claim 3, $x_i, i = 1, 2, 3$, has at most one nonadjacent vertex in $V(G) \setminus \{y\}$. Therefore, M has at most one vertex. If M has a vertex u, then $N(x)$ induces a triangle. We claim that there is at most one edge absent in $\langle N_2(x) \rangle$. On the contrary, suppose there are at least two edges absent in $\langle N_2(x) \rangle$. Recall that $|N_2(x) \setminus N(y)| \leq 1$ and $|N_2(x) \cap N(y)| \leq 3$. There exists a vertex $v \in N_2(x) \cap N(y)$ such that $d_{N_2(x)}(v) \leq |N_2(x)| - 2$. That is, v has at least three nonadjacent vertices in G (one is x, the other two are in $N_2(x)$). This contradicts $d(v) \geq n - 3$. It follows that $G' = \langle N(x) \cup N_2(x) \rangle$ is K_m or K_m^- with $m \geq 5$, and so G is Z_3-connected.

Finally we consider the case $M = \emptyset$, which implies $N_2(x) \supseteq N(y)$. Since $d(x_i) \geq n-3$ and $d(y_i) \geq n-3$ for each $i = 1, 2, 3$, we see that $\langle N(x)\rangle$ and $\langle N(y)\rangle$ each contains a path of length two. If there exists $u \in N(x) \cup N(y)$, say $u = x_1$ such that $e(x_1, N(y)) = 3$, then the subgraph induced by $\{x_1, y_1, y_2, y_3, y\}$ contains a W_4, and so G is Z_3-connected. So assume such a vertex does not exist. Hence $N_2(x) = N(y)$. Again by the degree condition, $N(x)$ and $N(y)$ each induces a triangle. Furthermore, we may assume that $E(N(x), N(y)) = \{y_1x_1, y_1x_2, y_2x_1, y_2x_3, y_3x_2, y_3x_3\}$. It is not difficult to check that $G^* = \langle N(x) \cup N(y)\rangle$ is triangularly connected and $\delta(G^*) = 4$. By Lemma 2.4, G^* is Z_3-connected, so is G. □

4. Proof of Theorem 1.7

Let G be a simple connected graph satisfying the condition of Theorem 1.7. By Lemma 3.2, it is suffices to prove that except for the 15 graphs described in Fig. 1, G is Z_3-connected. We claim that $\delta(G) \geq 2$. Otherwise, let u be the vertex of G which has only one neighbor, say v. Since G is connected, let w be another neighbor of v. Then $dis_G(u, w) = 2$, and so $d(w) \geq n-1$. This means that w is adjacent to any vertices of G including u, a contradiction. Therefore, $\delta(G) \geq 2$.

Let $\delta(G) = 2$ and $N(u) = \{u_1, u_2\}$. If $n \leq 4$, then it is easy to see that G is one of $G_1 \sim G_4$. So assume $n \geq 5$. Then $d(v) \geq n-2$ for any $v \in N_2(u)$, that is v is adjacent to any vertex in $V(G)\setminus\{u\}$. It follows that $V(G) = \{u\} \cup N(u) \cup N_2(u) \cup N_3(u)$. We claim $|N_2(u)| \geq 2$. Otherwise, let v be the only vertex of $N_2(u)$. Since $d(u_1) \leq 3$ and $dis_G(w, u_1) = 2$ for any $w \in N_3(u)$, we have $d(w) \geq n-3$. This contradicts $e(u, \{u, u_1, u_2\}) = 0$. If $|N_2(u)| = 2$, then $d(w) \geq n-4$ for any $w \in N_3(u)$. Thus $G' = \langle N_2(u) \cup N_3(u)\rangle$ is K_m with $m = |N_2(u) \cup N_3(u)|$. Then G is G_5 or G_6 if $m = 2$, and G is G_{13}(or G_{14}) if $m = 3$(or $m = 4$). Otherwise, G' is K_m with $m \geq 5$. Since $e(u, N(u) \cup G') = 2$ and $e(u_i, G') \geq 2$ for each $i = 1, 2$, by Lemma 2.2 and recursively using Observation 3.1, we see that G is Z_3-connected. If $|N_2(u)| \geq 3$, then $G' = \langle N_2(u) \cup N(u)\rangle$ is K_m or K_m^- with $m \geq 5$. As $e(x, G') \geq 2$ for any $x \in V(G) \setminus V(G')$, by Lemma 2.2 and Observation 3.1, G is Z_3-connected. So $\delta(G) \geq 3$.

We prove the main theorem by induction on $|V(G)|$. If $4 \leq n \leq 5$ and $\delta(G) = n-1$, then G is either G_4 or K_5 and by Lemma 2.2, G is Z_3 connected. If $n = 5$ and $\delta(G) = 3$, let x be the vertex with $N(x) = \{x_1, x_2, x_3\}$ and y be the remaining vertex of G. Then $\delta(\langle N(x)\rangle) \geq 1$, and so $\langle N(x)\rangle$ contains a path of length 2, say $x_1x_2x_3$. It follows by $N(y) = N(x)$ that $G \cong W_4$. By Lemma 2.2, G is Z_3-connected. Let $n \geq 6$. Suppose that G is a simple graph with minimized vertices and the theorem holds for G' with $|V(G')| < n$. By Lemmas 3.4 and 3.5, we further assume that G contains a K_4^-, which is the union of two triangles xyz and xyw with xy in common and $d(z) \geq 4$. Let $G' = G_{[zx,zy]}$. Then G' contains a Z_3-connected subgraph (2-circuit xyx). Let H be the maximal, Z_3-connected, connected subgraph of G' and $G^* = G'/H$. Denote by u^* the new vertex into which H is contracted. Then G^* is a connected graph. Suppose otherwise that G^* is not connected. It follows that z is a cut vertex and $\{zx, zy\}$ is an edge cut. Let G_1 and G_2 be two subgraphs of G such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{z\}$ and $x, y \in V(G_2)$. Since $d(z) \geq 4$, there is a vertex $z_1 \in V(G_1)$ such that $z_1 \in N(z)$. Then $d(z_1, y) = 2$. By the degree condition, $d(z_1) + d(y) \geq n$. On the other hand, $d(z_1) \leq |V(G_1)| - 1$ and $d(y) \leq |V(G_2)| - 1$, which implies that $d(z_1) + d(y) \leq |V(G_1)| + |V(G_2)| - 2 = n-1$. This contradiction proves that G^* is connected. Let $S = G - V(H)$. We have the following observations.

$$d_{G^*}(u) = d(u) \quad \text{for any } u \in V(G^*) \setminus \{u^*, z\},\ d_{G^*}(z) = d(z) - 2, \tag{4}$$

$$d_{G^*}(u^*) \geq d(t) - (|V(H)| - 1) + e(H - t, S) \quad \text{for any } t \in V(H), \tag{5}$$

and

$$e(v, H) \leq 1 \quad \text{for any } v \in V(G) \setminus V(H). \tag{6}$$

Since $\{x, y, w\} \subseteq V(H)$, we have $n^* = |V(G^*)| \leq n-2$. We are going to show that $d_{G^*}(u) + d_{G^*}(v) \geq n^*$ for any pair of vertices u, v of G^* with $dis_{G^*}(u, v) = 2$.

Let u, v be any pair of vertices of G^* with $dis_{G^*}(u, v) = 2$. If $\{u, v\} = \{u^*, z\}$, then assume $u = u^*$ and $v = z$; if $|\{u, v\} \cap \{u^*, z\}| = 1$, then assume $u = u^*$ or $v = z$. The following two cases occur.

Case 1. $u \neq u^*$.

If $dis_G(u, v) = 2$, then by (4), we have that $d_{G^*}(u) + d_{G^*}(v) \geq d(u) + d(v) - 2 \geq n-2 \geq n^*$. So $dis_G(u, v) \geq 3$ and suppose $d_{G^*}(u) + d_{G^*}(v) \leq n^* - 1$. Let u', v' be the vertices in H such that $dis_G(u, u') = 2$ and $dis_G(v, v') = 2$. We claim that we may choose u', v' such that $u' \neq v'$. On the contrary, $u' = v'$. Let $u'' \in V(H)$ and $v'' \in V(H)$ be the common neighbors of u, u' and v, v', respectively. Since $dis_G(u'', v'') = 2$, we have $|N(u'') \cap N(v'')| \geq n - (n-2) = 2$. By (6), $N(u'') \cap N(v'') \subseteq V(H)$. Let $\{u', v'\} \subseteq N(u'') \cap N(v'')$ with $u' \neq v'$, we are done.

Since $dis_G(u, u') = 2$ and $dis_G(v, v') = 2$, $d(u) + d(u') + d(v) + d(v') \geq 2n$. By (4), we have $d(u') + d(v') \geq 2n - (d_{G^*}(u) + d_{G^*}(v) + 2) \geq n + (n - n^* - 1)$. Hence $|N(u') \cap N(v')| \geq n - n^* - 1$. It follows by (6) and $|V(H)| = n - n^* + 1$ that $V(H) = N[u'] \cap N[v']$ and $N(u') \cup N(v') = V(G)$. So $uv', u'v \in E$ and $dis_G(t, u) = 2$, $dis_G(t, v) = 2$ for each vertex $t \in V(H) \setminus \{u', v'\}$. Let t_1, t_2 be any two distinct vertices in $V(H)$. Similarly, $d(t_1) + d(t_2) \geq n + (n - n^* - 1)$, $V(H) = N[t_1] \cap N[t_2]$ and $N(t_1) \cup N(t_2) = V(G)$. Since $|V(H)| \geq 3$, by the pigeonhole principle, each vertex in S has at least two neighbors in $V(H)$, this contradicts (6).

Case 2. $u = u^*$.

In this case, there must exist $t \in V(H)$ such that $dis_G(t, v) = 2$, and so $d(t) + d(v) \geq n$. By (4) and (5), the following inequality holds

$$d_{G^*}(u^*) + d_{G^*}(z) \geq d(t) + d(z) - 2 - (|V(H)| - 1) + e(H - t, S) \geq n^* - 2 + e(H - t, S).$$

If $e(H - t, S) \geq 2$, we are done. So assume that

$$e(H - t, S) \leq 1. \tag{7}$$

By the definition of H, any vertex of H must have at least two neighbors in H. Let $\{t_1, t_2\} \subseteq N(t) \cap V(H)$. Then $dis_G(a, t_1) = dis_G(a, t_2) = 2$, where a is the common neighbor of t and v. By (7), we assume $e(t_1, S) = 0$. So $n \leq d(a) + d(t_1) \leq |S| + |V(H)| - 1 = n - 1$, a contradiction.

We have shown that $d_{G^*}(u) + d_{G^*}(v) \geq n^*$ for any pair of vertices u, v of G^* with $dis_{G^*}(u, v) = 2$. By the induction hypothesis, either G^* is Z_3-connected or G^* is one of the 15 graphs in Fig. 1. If the former holds, we are done. So the latter holds. Since G^* has at most two vertices of degree two, G^* cannot be G_1, G_2. If G^* is G_3, then the two vertices of degree two must be z and u^*, and so G has at least two vertices of degree three. By Lemma 3.5, G is either Z_3-connected or one of $G_7 \sim G_{15}$ in Fig. 1. For $G_4, G_5, G_7 \sim G_{12}$, and G_{15}, they all contain at least four vertices of degree three which means G contains at least two vertices of degree three. By Lemma 3.5 again, G is either Z_3-connected or one of $G_7 \sim G_{15}$. Since G_6 comes from G_5 by adding one edge, if G^* is G_6, then we are done. If G^* is G_{13}, then the two vertices of degree two must be z, u^*, for any vertex u with $dis_G(u, z) = 2$, we have $n \leq d(u) + d(z) = 8$, and so $n = 8$. Therefore, $V(H) = \{x, y, w\}$. It follows by $\delta(G) \geq 3$ that $e(w, S) \geq 1$. If $e(w, S) = 2$, then x, y are two vertices of degree three, and so G is Z_3-connected. So $e(w, S) = 1$ and assume $e(x, S) = 0$. In this case, x, w are two vertices of degree three, we are done. Finally, if G^* is G_{14}, then the vertex of degree two must be one of z, u^*, say z (the case $d_{G^*}(u^*) = 2$ is similar). If u^* is a vertex of degree four, then we are done. So assume that the vertex of degree three must be u^* and let v be the other vertex of degree three. For any u with $dis_G(u, v) = 2$, we have $n \leq d(u) + d(v) = 7$. Hence $n = 7$, this contradicts $7 = n^* \leq n - 2$, and thus proves the theorem. □

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